

Perpendicular Bisector Quadrilaterals as a Substantial Learning Environment

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In the following we will present some learning activities for students which arise from questions and phenomena about constructing perpendicular bisectors of the sides of quadrilaterals, forming another quadrilateral, a so-called *perpendicular bisector quadrilateral*. Altogether this can be viewed as a substantial learning environment in geometry because students deal with motivating and challenging questions in a way which is typical of the process of doing mathematics. When choosing problems for students to investigate with dynamic geometry software, they should be interesting and challenging, but not beyond their reach. We believe the following set of problems provides a novel extension of the high school curriculum that should be accessible to most high school students. Two of the questions we will deal with here might be original as we have been unable to find any reference to them in the literature. But, as we will see, giving only small hints, the corresponding proofs are accessible for students.

1 Some well-known properties

First, let us define the term *perpendicular bisector quadrilateral*, for brevity we will denote them as *PB-quadrilaterals*, see also Grinberg (no date). Let $ABCD$ be a non-cyclic quadrilateral (Note 1). The perpendicular bisectors of the edges DA and AB meet at A' , the perpendicular bisectors of AB and BC meet at B' , the perpendicular bisectors of BC and CD meet at C' , the perpendicular bisectors of CD and DA meet at D' . Then the quadrilateral $A'B'C'D'$ will be called the *PB-quadrilateral* of the quadrilateral $ABCD$ (see Figure 1). Note that $ABCD$ and $A'B'C'D'$ have *different orientation* for convex quadrilaterals $ABCD$.

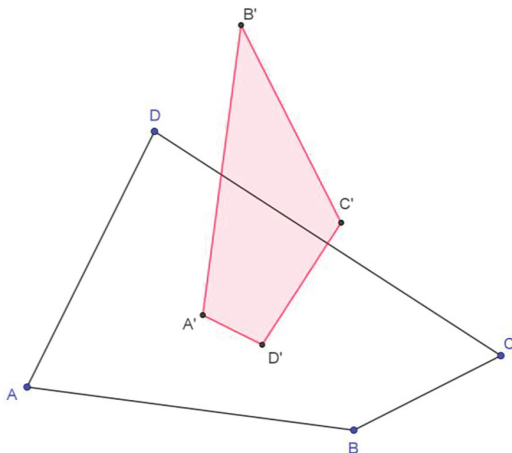


Figure 1

Below we give a list of some well-known properties with some hints for proofs in brackets (Note 2). The proofs are established by confronting students with tasks, namely exploring the phenomena experimentally with dynamic geometry software, and subsequently explaining why they are true (Note 3).

1. The points A', B', C', D' are the circumcentres of $\triangle ABD$, $\triangle ABC$, $\triangle BDC$, $\triangle ACD$, respectively.

[Since A' is the intersection point of the perpendicular bisectors of AB and AD , it immediately follows that A' is the circumcentre of $\triangle ABD$, etc.]

2. The sides of $A'B'C'D'$ are perpendicular to the corresponding sides of $ABCD$ (more precisely: $A'B' \perp AB$, etc.).

[A' and B' must lie on the perpendicular bisector of AB , why? Analogous for the other sides.]

3. The interior angles of $ABCD$ are supplementary to the corresponding interior angles of $A'B'C'D'$ (meaning $\angle A$ and $\angle A'$ are supplementary, etc.).

[Let the extension of $B'A'$ intersect AB at E , and the extension of $D'A'$ intersect AD at F . What kind of quadrilateral is $AEA'F$? What does that mean for $\angle A'$? Why? See Figures 2a]

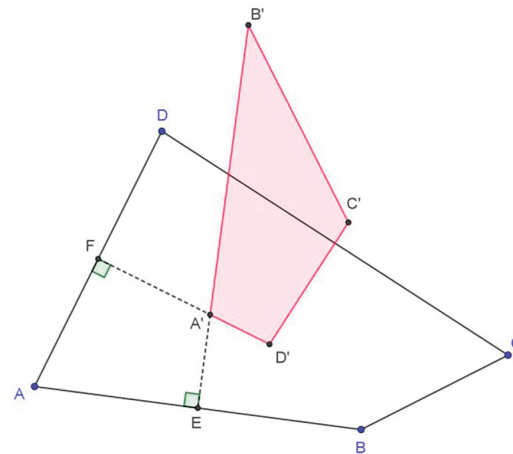


Figure 2a: To property 3

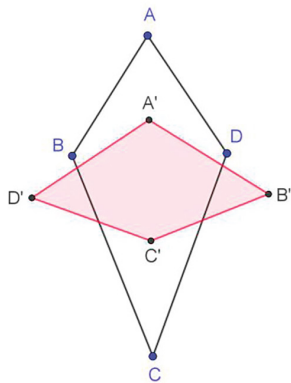


Figure 2b: To Property 7

- The diagonals of $A'B'C'D'$ lie on perpendicular bisectors of the *non-corresponding* diagonals of $ABCD$ (more precisely: line $A'C'$ is the perpendicular bisector of diagonal BD and line $B'D'$ is the perpendicular bisector of diagonal AC).

[What can be said about the distances of A' and C' to B and D , respectively? What does that mean for the diagonals $A'C'$ and BD ? Analogous with the other pair of *non-corresponding* diagonals?]

- The intersection point of the diagonals of the PB-quadrilateral $A'B'C'D'$ is equidistant from each pair of opposite vertices of the original quadrilateral $ABCD$, thus, can be called a quasi-circumcentre, see: <http://dynamicmathematicslearning.com/quasi-circumcentre-quad.html>

[How is this property connected to the previous one?]

- The diagonals of $ABCD$ and $A'B'C'D'$ make equal angles.

[How is this property connected to the previous two?]

- PB-quadrilateral $A'B'C'D'$ is a kite if, and only if, $ABCD$ is a kite (Figure 2b).

[How is this property connected to properties 3 and 6?]

Such challenges can often be formulated also in another way, namely not telling the result and just asking for a proof but asking questions and giving room for conjectures. As an example, for property (6): "What do you notice about the angles between the diagonals in the two quadrilaterals $ABCD$ and $A'B'C'D'$? Give an explanation why your conjecture is holds".

New mathematics is often created by asking *what if* questions which is a valuable problem posing skill, see e.g. Humenberger (2025) or Papadopoulos et al. (2020). Students should be encouraged to ask such questions. Helping them to deal with such questions can positively influence students' beliefs that mathematics is neither static nor boring, see e.g. Mason et al. (2010). Such questions can readily be explored by students using

dynamic geometry to experimentally see for themselves what happens, before thinking about arguments and proofs.

An illustrative example about asking a *what if* question about PB-quadrilaterals might be here to ask: What type of PB-quadrilateral $A'B'C'D'$ is formed if $ABCD$ is a parallelogram? Perhaps to their surprise they might discover using dynamic geometry that not only is $D'C'BA'$ (rather obviously) a parallelogram, but more importantly, it is also similar to $ABCD$. Assisting students to prove it and reflect carefully on the proof might lead them to realize that the result can even be generalized to a trapezoid. Alternatively, one can suggest another *what if* question which seems natural to ask: What type of PB-quadrilateral $D'C'BA'$ is formed if $ABCD$ is a trapezoid? Either way, this would lead to the following interesting result.

- PB-quadrilateral $D'C'BA'$ is similar to $ABCD$ if $ABCD$ is a trapezoid.

Short proof (Figure 3): Let $AB \parallel CD$. From property 2 we have $AD \perp A'D'$ and $AB \perp A'B'$; from property 4 we have $AC \perp B'D'$, and in addition, because $AB \parallel CD$, we have $DC \perp A'B'$. Thus, $\triangle ACD \sim \triangle D'B'A'$ with factor, say k . In the same manner one can establish $\triangle ACD \sim \triangle D'B'C'$ with the same factor k (note that $B'D' = kAC$ is involved in both cases).

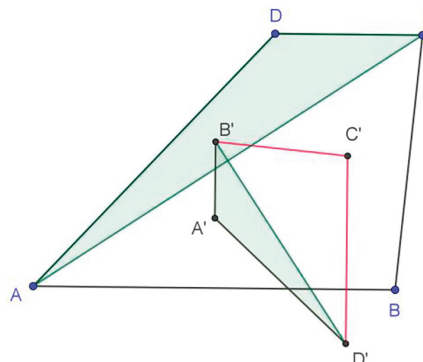


Figure 3

Not necessarily for proving in class, as the proof is a bit more advanced, but quite suitable for exploring with dynamic geometry we can ask another *what if* question: What type of PB-quadrilateral $A'B'C'D'$ is formed if $ABCD$ is a tangential quadrilateral?

PB-quadrilateral $A'B'C'D'$ is tangential (circumscribed) if and only if $ABCD$ is tangential (circumscribed) – see e.g. Grinberg, for the "if" part also De Villiers (1994, pp. 192-193) or Hauke (2019). For an online dynamic version see: <https://dynamicmathematicslearning.com/circumquad-perpbisector.html>

2 Two (possibly) new properties

Now we come to seemingly new properties of PB-quadrilaterals. And the surprising fact here is that

students can also manage the proofs, because the proofs are straight forward. So, why might these be new results? Probably because nobody asked the corresponding question up to now, surely not because nobody was able to manage a proof. And this is really a special situation that does not often occur in mathematics education: Students can deal with results that cannot be found in the literature! This may well be particularly motivating for students. We formulate our findings as theorems.

Theorem 1 The four triangles in which $ABCD$ is divided by its diagonals are similar to the four corresponding triangles in which $A'B'C'D'$ is divided by its diagonals (Figure 4, similar triangles coloured correspondingly).

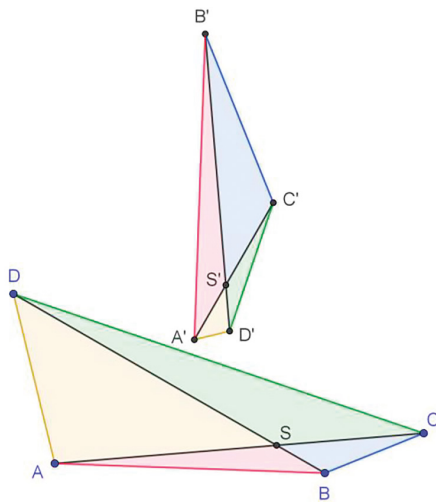


Figure 4

Proof It is easy to see the similarities of corresponding triangles because they have the same angles. Since $A'S' \perp DS$, $A'D' \perp DA$ and $S'D' \perp AS$, we can conclude $\angle ADS = \angle D'A'S'$, $\angle DSA = \angle A'S'D'$, $\angle SAD = \angle S'D'A'$, telling us that $\triangle DAS \sim \triangle A'D'S'$. Likewise, we get the other similarities.

So, both quadrilaterals consist of four pairwise similar triangles having a common vertex S and S' which are the respective intersection points of the diagonals. Note that in the pairs of similar triangles the vertices A, B, C, D and A', B', C', D' have to be interchanged, more precisely: in the similar triangles $\triangle ABS \sim \triangle B'A'S'$ we have S corresponds to S' but the vertex A corresponds to B' and B to A' ; analogously with the other pairs of similar triangles.

Since the proof is, technically seen, not difficult it can surely enrich the learning environment for students by providing a novel, but accessible challenge to them. Students could be given the following task: Construct with dynamic geometry software a quadrilateral $ABCD$ and its PB-quadrilateral $A'B'C'D'$. Look, in both cases, at the four triangles into which the diagonals divide them and compare them; what do you notice? [possible hint: look at angles and ratios!] Can you give a proof for your conjecture?

A surprising and motivating aspect of this problem: While it can probably be solved by students with the aid of a small hint, the result seems to be a new theorem – at least we did not find a reference for it (unless well hidden in the literature?). The possible novelty of the theorem is here not so important, much more important here is that it can be tackled by students. This can provide positive and motivating experiences for them in doing mathematics as a process, so that they can feel a bit like researchers, discovering and proving new things.

While Theorem 1 is interesting for itself, it also serves as a lemma for Theorem 2 below, since its proof will rely primarily on the similar triangles of Theorem 1.

Let us next examine the PB-quadrilateral of an Apollonius quadrilateral. It is a quadrilateral with the property $a \times c = b \times d$; in other words, it has equal products of opposite sides; see Haruki & Rassias (1998), and also: https://en.wikipedia.org/wiki/Apollonius_quadrilateral

Analogously to section 1 where the perpendicular bisectors of a tangential (circumscribed) quadrilateral produce another tangential (circumscribed) quadrilateral, the same also holds for Apollonius quadrilaterals. Let us formulate this as a theorem, which we ourselves experimentally discovered by using dynamic geometry.

Theorem 2 The PB-quadrilateral $A'B'C'D'$ is an Apollonius quadrilateral if, and only if, $ABCD$ is an Apollonius quadrilateral. A dynamic online version illustrating Theorem 2 for an Apollonius quadrilateral (as well as Theorem 1) is available at: <http://dynamicmathematicslearning.com/perpendicular-bisectors-of-apolonius-quad.html>

Proof We will restrict the proof to the case of convex quadrilaterals $ABCD$ (Note 4). See Figure 5.

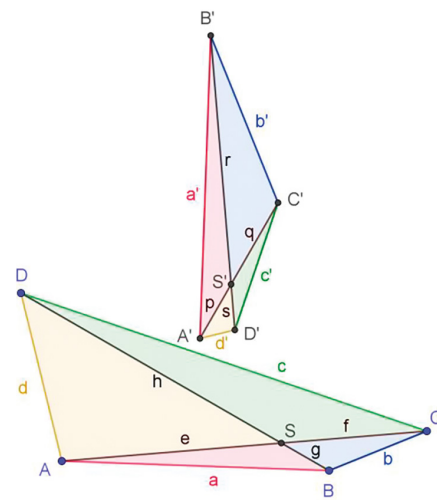


Figure 5

By the triangle similarities mentioned above we have

$$a' = a \cdot \frac{p}{g}, b' = b \cdot \frac{q}{g},$$

$$c' = c \cdot \frac{q}{h}, d' = d \cdot \frac{p}{h}.$$

Thus, we can conclude our claim – in both directions! – in a surprisingly easy way:

$A'B'C'D'$ is an Apollonius quadrilateral

$$\Leftrightarrow a' \cdot c' = b' \cdot d'$$

$$\Leftrightarrow (a \cdot \frac{p}{g})(c \cdot \frac{q}{h}) = (b \cdot \frac{q}{g})(d \cdot \frac{p}{h})$$

$$\Leftrightarrow a \cdot c = b \cdot d$$

$\Leftrightarrow ABCD$ is an Apollonius quadrilateral.

Here, again, a possible formulation of the problem for students: In geometry, a special type of quadrilateral is called an Apollonius quadrilateral. Look up the definition on the internet. Construct with a dynamic geometry software an Apollonius quadrilateral $ABCD$ and its PB-quadrilateral $A'B'C'D'$. What do you notice? Is it also an Apollonius quadrilateral? Explore experimentally with dynamic geometry and formulate a conjecture. Can you give a proof for your conjecture?

Again, as was the case with Theorem 1, students can establish a proof for Theorem 2 autonomously because they only need to use Theorem 1 and express the new lengths a', b', c', d' .

Conclusion

We presented the topic *Perpendicular Bisector Quadrilaterals* as a substantial and creative learning environment for students. They can explore situations with dynamic geometry, make conjectures, and finally establish proofs, thereby participating in the process of doing mathematics. All the proofs are straightforward and elementary, thus suitable for high school students, especially when working in groups. Nobody needs to be ingenious, although partially dealing with facts which seem to be not mentioned in the literature.

So, altogether, this topic has high educational potential and can enrich the teaching process at several levels. It is noteworthy that we have here an authentic example where students can feel like researchers themselves. The possible novelty of some of the results is less important here than the educational and motivational aspects. Finally, let us stress once again the value of dealing with *what if* questions in mathematics and in mathematics education, in order that students get more engaged in doing mathematics themselves and thus changing their beliefs about what mathematics is about.

Notes

1. Throughout the paper we will assume that $ABCD$ is not cyclic because otherwise $A'B'C'D'$ would degenerate to a single point, namely the circumcentre.

2. See also https://en.wikipedia.org/wiki/Perpendicular_bisector_construction_of_a_quadrilateral
3. The proofs of properties 1-7 are straightforward, and thus, seem suitable for high school students. Concerning several functions of proof see e.g. De Villiers (2012).
4. If students know the Apollonius circle – the other cases (concave, crossed) work analogously – one can also witness this experimentally in a dynamic way using dynamic geometry software: Given A, B, C fixed, construct the Apollonius circle with regard to the points A and C and the ratio $AB : BC$ (of course, B lies on this circle); choose an arbitrary point $D \neq B$ on this circle and we have an Apollonius quadrilateral $ABCD$. Then construct the corresponding PB-quadrilateral $A'B'C'D'$ and observe that $a' \cdot c' - b' \cdot d' = 0$ always holds when dragging D on the Apollonius circle.

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